ME 4555 - Lecture 12 - More Laplace Transforms

Recall: the Laplace transform is a technique/tool for solving the equations of motion that describe an LTI (linear time-invariant) system. These are linear ODEs.

Some examples

$$f(t) \underset{\mathcal{L}^{-1}\{\cdot\}}{\overset{\mathcal{L}\{\cdot\}}{\rightleftarrows}} F(s)$$

$f(t)$	$F(s)$	
$H(t)$	$\frac{1}{s}$	
$af(t)$	$aF(s)$	linearity
$f(t)+g(t)$	$F(s)+G(s)$	linearity
$\frac{df(t)}{dt}$	$sF(s) - f(0)$	
$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0) - f'(0)$	
e^{-at}	$\frac{1}{s+a}$	
$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$	

definition:

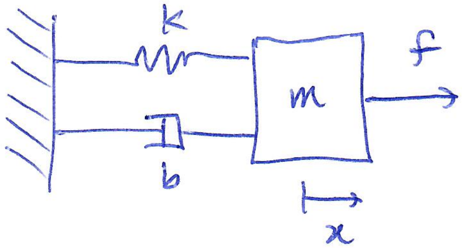
$$F(s) \equiv \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

$$\begin{pmatrix}\text{System EoM}\\ \text{(ODEs)}\\ \text{in variable } x(t)\end{pmatrix} \xrightarrow{\mathcal{L}} \begin{pmatrix}\text{Algebraic}\\ \text{Equations}\\ \text{in variable } X(s)\end{pmatrix}$$

$$\Big\downarrow \text{ solve for } X(s)$$

$$\begin{pmatrix}\text{obtain solution}\\ x(t) = \ldots\end{pmatrix} \xleftarrow{\mathcal{L}^{-1}} \begin{pmatrix}\text{obtain solution}\\ X(s) = \ldots\end{pmatrix}$$

Ex : spring - mass - damper.



EOM: $m\ddot{x} + b\dot{x} + kx = f$

Take Laplace transform, $x(t) \xrightarrow{\mathcal{L}} X(s)$ and $f(t) \xrightarrow{\mathcal{L}} F(s)$.

$$m\left(s^2X - sx(0) - \dot{x}(0)\right) + b\left(sX - x(0)\right) + kX = F$$

$$\Rightarrow (ms^2 + bs + k)X = F + (ms+b)x(0) + m\dot{x}(0)$$

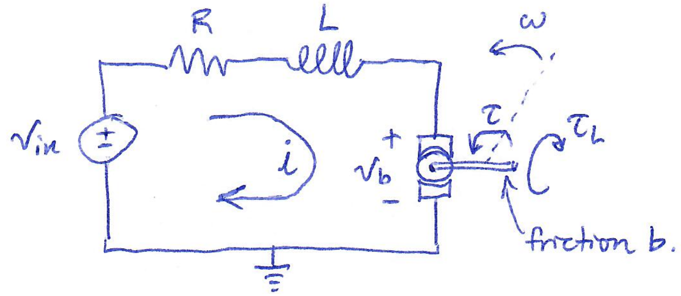
$$\Rightarrow \underbrace{X(s)}_{\text{response}} = \underbrace{\underbrace{\left(\frac{1}{ms^2+bs+k}\right)}_{\text{transfer function}} \underbrace{F(s)}_{\text{input}}}_{\text{particular solution (depends on input)}} + \underbrace{\left(\frac{(ms+b)x(0) + m\dot{x}(0)}{ms^2+bs+k.}\right)}_{\text{homogeneous solution, depends on initial conditions } x(0),\ \dot{x}(0) \text{ (initial position + velocity of mass)}}$$

the particular solution solves $m\ddot{x} + b\dot{x} + kx = f$ when $x(0) = \dot{x}(0) = 0$.

the homogeneous solution solves $m\ddot{x} + b\dot{x} + kx = 0$ for given $x(0), \dot{x}(0)$.

the transfer function $G(s) = \dfrac{X(s)}{F(s)} = \dfrac{1}{ms^2+bs+k}$ tells us how the system responds to input (when initial conditions are zero).

Ex: DC motor.



GOAL: Find $\omega(t)$ in terms of $v_{in}(t)$

time-domain

(1) $v_{in}(t) - v_b(t) = R\,i(t) + L\frac{di(t)}{dt}$

(2) $\tau(t) = K_t\, i(t)$

(3) $J\dot{\omega}(t) = \tau(t) - b\,\omega(t) - \tau_L(t)$

(4) $v_b(t) = K_b\,\omega(t)$

sub (2) → (3)
and (4) → (1)

(5) $v_{in}(t) = R\,i(t) + L\frac{di(t)}{dt} + K_b\,\omega(t)$

(6) $-\tau_L(t) = J\dot{\omega}(t) + b\,\omega(t) - K_t\,i(t)$

solve (6) for $i(t)$

(7) $i(t) = \frac{1}{K_t}\left[J\dot{\omega}(t) + b\,\omega(t) + \tau_L(t)\right]$

substitute (7) into (5)

(8) $\boxed{\frac{LJ}{K_t}\ddot{\omega} + \left(\frac{RJ+Lb}{K_t}\right)\dot{\omega} + \left(\frac{Rb}{K_t} + K_b\right)\omega = v_{in} - \frac{L}{K_t}\dot{\tau}_L - \frac{R}{K_t}\tau_L}$

s-domain (take Laplace transform)

$\rightarrow V_{in}(s) - V_b(s) = R\,I(s) + L\left(sI(s) - i(0)\right)$

$\rightarrow T(s) = K_t\,I(s)$

$\rightarrow J\left(s\Omega(s) - \omega(0)\right) = T(s) - b\,\Omega(s) - T_L(s)$

$\rightarrow V_b(s) = K_b\,\Omega(s)$

set initial conditions to zero
$i(0) = 0,\ \omega(0) = 0.$

$$\begin{cases} V_{in} - V_b = (R + Ls)I \\ T = K_t I \\ T - T_L = (b + Js)\Omega \\ V_b = K_b\Omega \end{cases}$$

(continued on next page.

$$\begin{aligned}
&(1) & V_{in} - V_b &= (Ls + R)\, I \\
&(2) & T &= K_t\, I \\
&(3) & T - T_L &= (Js + b)\, \Omega \\
&(4) & V_b &= K_b\, \Omega
\end{aligned}$$

DC motor equations in the s-domain.

inputs: $\{V_{in}(s), T_L(s)\}$, states: $\{I(s), \Omega(s)\}$, intermediate quantities: $\{T(s), V_b(s)\}$

We want to relate $V_{in}(s)$ and $\Omega(s)$ so we must eliminate $\{I(s), V_b(s), T(s)\}$

substitute $(2) \rightarrow (3)$ and $(4) \rightarrow (1)$.

$$(5) \quad V_{in} = (Ls + R)\, I + K_b\, \Omega$$

$$(6) \quad -T_L = (Js + b)\, \Omega - K_t\, I$$

Solve for I in (6): $I = \left(\frac{Js + b}{K_t}\right)\Omega + \frac{1}{K_t} T_L$ (7).

substitute this into (5):

$$(8) \quad V_{in} = (Ls + R)\left[\left(\frac{Js + b}{K_t}\right)\Omega + \frac{1}{K_t} T_L\right] + K_b\, \Omega.$$

Expand and rearrange: {same as (8) on prev. page!}

$$\underbrace{\left(\frac{LJ}{K_t} s^2 + \left(\frac{RJ + Lb}{K_t}\right) s^2 + \left(\frac{Rb}{K_t} + K_b\right)\right)}_{H(s)} \Omega = V_{in} - \left(\frac{L}{K_t} s + \frac{R}{K_t}\right) T_L$$

Transfer function from $V_{in} \rightarrow \Omega$: $\dfrac{\Omega}{V_{in}} = \dfrac{1}{H(s)}$ {what you get if $T_L = 0$ and V_{in} is only input.}

Transfer function from $T_L \rightarrow \Omega$: $\dfrac{\Omega}{T_L} = \dfrac{Ls + R}{K_t H(s)}$ {what you get if $V_{in} = 0$ and T_L is only input.}

Ex A general linear ODE. input $u(t)$, output $y(t)$.

$$y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \dots + a_1 \dot{y}(t) + a_0 y(t) = b_{n-1} u^{(n-1)}(t) + \dots + b_1 \dot{u}(t) + b_0 u(t)$$

$\downarrow$ take Laplace transform with zero initial conditions.

$$(s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0) Y(s) = (b_{n-1} s^{n-1} + \dots + b_1 s + b_0) U(s)$$

$\downarrow$ re arrange.

Transfer function: $\boxed{G(s) = \dfrac{Y(s)}{U(s)} = \dfrac{b_{n-1} s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}}$

☆ the transfer function for an LTI system is always a rational function of s (ratio of two polynomials)

☆ the degree of the denominator (n) is called the <u>order</u> of the system. So $G(s)$ above is an n^{th} order system.

☆ the order of a system is equal to the number of integrators needed to represent it in Simulink. i.e. a spring-mass-damper system is a 2^{nd} order system because its transfer function is $\dfrac{1}{ms^2 + bs + k}$ and the denominator has degree 2.